Covariant scalar representation of $iosp(d, 2/2)$ and quantization of the scalar relativistic particle

1996 J. Phys. A: Math. Gen. 29 1245

(http://iopscience.iop.org/0305-4470/29/6/011)

J. Phys. A: Math. Gen. **29** (1996) 1245–1260. Printed in the UK

Covariant scalar representation of $iosp(d, 2/2)$ and quantization of the scalar relativistic particle

P D Jarvis and I Tsohantjis
Department of Physics, University of Tasmania, GPO Box 252C Hobart, Australia 7001

Received 26 June 1995, in final form 22 November 1995

Abstract. A covariant scalar representation of $iosp(d, 2/2)$ is constructed and analysed in comparison with existing BFV-BRST methods for the quantization of the scalar relativistic particle. It is found that, with appropriately defined wavefunctions, this $iosp(d, 2/2)$ produced representation can be identified with the state space arising from the canonical BFV-BRST quantization of the modular-invariant, unoriented scalar particle (or antiparticle) with admissible gauge-fixing conditions. For this model, the cohomological determination of physical states can thus be obtained purely from the representation theory of the $iosp(d, 2/2)$ algebra.

1. Introduction and main results

The understanding of the quantization problem for systems with constraints has had a long development since the seminal monographs of Dirac [1]. The techniques introduced to handle gauge theories such as non-Abelian Yang–Mills–Shaw theory and (linearized) gravity culminated in the demonstration of global supersymmetries [2] for such systems, under which gauge and ghost degrees of freedom transform, and which also play a role even at the level of classical dynamics with finitely many degrees of freedom. In certain cases it is possible to unify further these 'quantization' supersymmetries with other symmetries possessed by the system, particularly those associated with the constraint algebra, so that the entire state space may be constructed from the representation theory of the enlarged algebra (see below). The ultimate goal of such work is that sufficient understanding of the gauge symmetries themselves, the nature of their graded extensions, and the associated representation theory, may enable admissible quantization(s) to be implemented systematically (and covariantly) at this algebraic level.

In the present paper, some preliminary steps in this direction are taken: the attitude adopted is that the general principles of this algebraic version of the quantization programme should emerge from detailed consideration of particular case studies. The initial example taken up below, is a quantum mechanical one, that of the scalar relativistic particle. In a forthcoming paper [3], it is intended to extend the analysis to the spinning particle. The enlarged algebra in these cases turns out to be an orthosymplectic extension of the Poincaré spacetime symmetry algebra. Subsequent papers in this series will consider other first quantized models, as well as second-quantized gauge field theories, for which the full structure of the extended algebra is not yet established.

Before proceeding to discuss the details of the paper and the main results, it is useful to give a brief historical review of the evolution of understanding of the nature of extended symmetries for constraint quantization. Following the introduction of scalar–vector spacetime supersymmetries in field theory in connection with critical systems [4] and with gauged internal superalgebras [5] the first presentations of BRST [2] and anti-BRST [6] transformations in superspace [7] were given a covariant $osp(d-1, 1/2)$ formulation for Yang–Mills–Shaw theory and gravity [8], in which the ghost fields were leading terms in superfield expansions of the graded components of the 'superpotential', and the BRST operators are supertranslations. Such formulations have recently been used in discussions of renormalization and Ward identities [9], and in discussions of higher derivative field theories [10].

The consistent description of classical and quantum Hamiltonian systems with constraints can be attempted primarily using the Dirac approach [1]. The simplest cases of relativistic point particles (scalar or spinning) have been intenslively investigated [11] resulting in a deeper understanding of the classical formulation of the problem and its quantization. One formulates the classical system in which only first-class constraints participate, describing a particle or an antiparticle leaving the problem of admissible gauge choices open for investigation. The path integral, although manifestly covariant is gauge dependent. Moreover after quantization one does not get a canonical gauge. On the other hand, using the standard known actions for point particles, digressing from Dirac's approach, one can choose a true canonical quantization method [12] (without imposing a manifest covariance and choosing the gauge from the beginning) one can describing a particle and antiparticle at the same time.

With the development of the BFV approach [13] to canonical quantization of systems with open gauge algebras arises the issue of extended quantization symmetries also in this context. Numerous works on the BFV-BRST quantization of the scalar relativistic particle exist in the literature. Following earlier analysis [14] on the compatibility of boundary conditions and gauge-fixing terms, it was shown [15, 16] that the action following from the BFV-BRST canonical analysis does indeed possess an extended spacetime supersymmetry, with respect to $iosp(d, 2/2)$; this was extended to the first quantization of the spinning particle, the galilean particle and the massless conformally-invariant particle [17] and also to the bosonic string [15]. More general approaches to covariant quantization and string field theory involving orthosymplectic spacetime supersymmetries have also been given [18–21]. Algebraic aspects of the BFV-BRST extended constraint algebra have been discussed in general, leading to the expectation that [22] $osp(1, 1/2)$ or [23] $igl(1/1)$ symmetries are always realized; the bosonic string would then be expected [22] to possess a quantization covariant with respect to $osp(26, 2/2)$.

In the present paper our aim is to give a detailed analysis of the extended $iosp(d, 2/2)$ spacetime quantization symmetry of the relativistic scalar particle in d dimensional Minkowski space. In recent work Cornwell and Hartley [24, 25] have developed formal aspects of the representation theory of orthosymplectic superalgebras, and this forms the basis of our construction. Specifically, we develop (section 2 below) a certain massless (irreducible) covariant scalar produced algebra module. This is then compared (section 3) with the state space arising from the quantization of the scalar relativistic particle, following the detailed analyses of Govaerts [28]. After appropriate canonical transformations of variables, and identification of wavefunctions, the respective algebra actions are shown to be homomorphic. Concluding remarks and an outlook for further work are given in section 4 below.

The major result of our analysis is thus that the quantization and (cohomological) identification of physical sates can be obtained for this model, purely from the representation theory of the $iosp(d, 2/2)$ algebra. In concluding this introduction, it should be pointed out that our approach does not require a superfield formalism (Grassmann variables arise only as dynamical degrees of freedom at the classical level in the BFV method), the produced algebra representations being developed explicitly in terms of appropriate multiplets of wavefunctions. Further, the $iosp(d, 2/2)$ covariance is shown directly for the state space, rather than via the derived phase or configuration space path integral representations, as has been shown in other approaches [16, 15]. In fact, issues of gauge invariance for physical states and their inner products certainly arise at the canonical level. As will be discussed further below, their resolution requires taking explicit account of Teichmüller space and modular invariance for this problem. The module homomorphism is between (one of two types of) produced $iosp(d, 2/2)$ representation, and, in technical terms [28], the BFV-BRST canonical quantization of the modular-invariant fundamental Hamiltonian description of the unoriented scalar relativistic particle (or antiparticle, respectively).

2. Representation theory of $iosp(d, 2/2)$

In this section we discuss those elements of the produced representation theory of inhomogeneous super-algebras [24, 25] which will be needed for our algebraic consruction of the particle quantization using the superalgebra $iosp(d, 2/2)$. The abstract theory of induced or produced representations for this case will be treated in a separate work.

Notation

The $iosp(d, 2/2)$ superalgebra is a generalization of $iso(d, 2)$. The metric tensor g of $iosp(d, 2/2)$ has a diagonal block form with the entries being the metric tensor of $so(d, 2)$ with -1 occurring d times, $g_{ab} = \mathrm{diag}(1, -1, \dots, -1, 1)$, and the symplectic metric tensor being given by $\epsilon_{12} = -\epsilon_{21} = i$ and $\epsilon^{\alpha\beta} = \epsilon_{\alpha\beta}$. Here latin indices take values $0, 1, \dots, d-1, d, d+1$, unless otherwise specified, and greek indices $\alpha, \beta, \dots$ take values 1,2, while $\lambda, \mu, \nu \dots$ take values $0, 1, \dots, d-1$ The homogeneous even subalgebra is $so(d, 2) \oplus sp(2, \mathbb{R})$. $so(d, 2)$ is generated by $J_{ab} = -J_{ba}$, and $sp(2, \mathbb{R})$ is generated by $K_{\alpha\beta} = K_{\beta\alpha}$. The odd generators will be denoted by $L_{a\alpha}$. The inhomogeneous part $i(d, 2/2)$ consists of $d+2$ even translations P_a in the $(d, 2)$ pseudo-Euclidean space, and two odd nilpotent translations Q_α. The generators can also be expressed in a light cone basis where we choose, for the coordinates, momenta and generators

$$\begin{aligned} x_\pm &= (1/\sqrt{2})(x_{d+1} \pm x_d) \\ P_\pm &= (1/\sqrt{2})(P_{d+1} \pm P_d) \\ J_{\pm a} &= (1/\sqrt{2})(J_{(d+1)a} \pm J_{(d)a}) \\ L_{\pm\alpha} &= (1/\sqrt{2})(L_{(d+1)\alpha} \pm L_{d\alpha})\,. \end{aligned} \tag{1}$$

Such a choice is not accidental, as will become apparent later. In this case latin indices $a, b = 0, 1, \dots, d-1, +, -$, while $g_{\mu\nu} = \mathrm{diag}(1, -1, \cdots -1)$ and $g_{+-} = g_{-+} = 1$. The non-zero $iosp(d, 2/2)$ commutation relations in the light cone choice read as follows [24, 25]:

$$[J_{ab}, J_{cd}] = -i(g_{ac}J_{bd} - g_{bc}J_{ad} + g_{bd}J_{ac} - g_{ad}J_{bc}) \tag{2a}$$

$$[K_{\alpha\beta}, K_{\gamma\delta}] = -(\epsilon_{\alpha\gamma}K_{\beta\delta} + \epsilon_{\beta\gamma}K_{\alpha\delta} + \epsilon_{\beta\delta}K_{\alpha\gamma} + \epsilon_{\alpha\delta}K_{\beta\gamma}) \tag{2b}$$

$$[J_{ab}, L_{c\alpha}] = -i(g_{ac}L_{b\alpha} - g_{bc}L_{a\alpha}) \qquad [K_{\alpha\beta}, L_{a\gamma}] = -(\epsilon_{\alpha\gamma}L_{a\beta} + \epsilon_{\beta\gamma}L_{a\alpha}) \tag{2c}$$

$$[L_{a\alpha}, L_{b\beta}] = i(\epsilon_{\alpha\beta}J_{ab} - ig_{ab}K_{\alpha\beta}) \tag{2d}$$

$$[J_{ab}, P_c] = -i(g_{ac}P_b - g_{bc}P_a) \qquad [K_{\alpha\beta}, Q_\gamma] = -(\epsilon_{\alpha\gamma}Q_\beta + \epsilon_{\beta\gamma}Q_\alpha) \tag{2e}$$

$$[L_{a\alpha}, P_c] = -ig_{ac}Q_\alpha \qquad [L_{a\alpha}, Q_\beta] = -i\epsilon_{\alpha\beta}P_a\,. \tag{2f}$$

It should be noted that the generators satisfying the above algebra are those of the complexification of $iosp(d, 2/2)$. That is, they are linearly independent over $\mathbb{R}$ and $\mathbb{C}$. Moreover they can be considered as a basis of $iosp(d + 2/2)$, the inhomogeneous extension of the compact real form of an appropriate basic classical simple complex Lie superalgebra† $osp(d, 2/2)$ is the non-compact real form of one of the basic classical simple complex Lie superalgebras $B(m, 1)$ or $D(m, 1)$ [26]. It can be obtained from an appropriate automorphism of the compact real forms of the above mentioned superalgebras [27]. A realization of $osp(d, 2/2)$ is provided by the $(d+4)$-dimensional supermatrices M satisfying

$$M^{\mathrm{st}}g - (-1)^{[M]}gM = 0$$

where g is the metric tensor (see footnote) and $[M]$ is 0 (for even supermatrices) or 1 (for odd supermatrices) respectively and 'st' denotes the supertranspose of the supermatrix M defined as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\mathrm{st}} = \begin{pmatrix} A^{\mathrm{t}} & -C^{\mathrm{t}} \\ B^{\mathrm{t}} & D^{\mathrm{t}} \end{pmatrix}$$

where t denotes the normal transpose of a matrix. The obvious quadratic Casimir operator (the analogue of the mass operator in the Poincaré case) is

$$C_2 = P_aP^a + Q_\alpha Q^\alpha. \tag{3}$$

A generalized Pauli–Loubanski operator has been found, and the fourth-order Casimir is given by

$$C_4 = \tfrac{1}{3}W_{abc}W^{abc} - W_{ab\alpha}W^{ab\alpha} + W_{a\alpha\beta}W^{a\alpha\beta} - \tfrac{1}{3}W_{\alpha\beta\gamma}W^{\alpha\beta\gamma} \tag{4}$$

where

$$\begin{aligned}
W_{abc} &= J_{ab}P_c + J_{bc}P_a + J_{ca}P_b \\
W_{ab\alpha} &= J_{ab}Q_\alpha + L_{a\alpha}P_b - L_{b\alpha}P_a \\
W_{a\alpha\beta} &= iL_{a\alpha}Q_\beta + K_{\alpha\beta}P_a + iL_{a\beta}Q_\alpha \\
W_{\alpha\beta\gamma} &= K_{\alpha\beta}Q_\gamma + K_{\beta\gamma}Q_\alpha + K_{\gamma\alpha}Q_\beta.
\end{aligned} \tag{5}$$

The covariant scalar multiplet

We now turn to the construction of the covariant scalar multiplet, adapting the exposition of Hartley and Cornwell [24, 25]. Let us start with the definition the covariant representations of the group $ISO(d, 2)$ which follows exactly the same lines of exposition as that of the normal Poincaré group. It should also be noted that, although not directly used, we should deal with the universal covering group of proper orthochronous $ISO_0(d, 2)$. The $d+2$ dimensional pseudo-Euclidean space is identified with the coset space $ISO(d, 2)/SO(d, 2)$. We shall denote a general element of $ISO(d, 2)$ by (t, Λ) where $(0, \Lambda)$ is a rotation

† Writing the basis elements which are linearly independent over $\mathbb{R}$, and thus form the real $iosp(d, 2/2)$, as $M_{ab} = iJ_{ab}$,$M_{\alpha\beta} = iK_{\alpha\beta}$, $M_{a\beta} =^{i\pi/4} L_{a\beta}$, $R_a = i\zeta P_a$, $R_\alpha = e^{i\pi/4}\zeta Q_\alpha$, the commutation relations read $[M_{AB}, M_{CD}] = C^{EF}_{AB,CD}M_{EF}$, $[M_{AB}, R_C] = C^{D}_{AB,C}R_D$, ζ being an arbitrary non-zero real constant, with capital Latin indices in the range $0, 1, \ldots, d+4$, $g_{(\alpha+d+2)(\beta+d+2)} = \epsilon_{\alpha\beta}$, and where the structure constants are built covariantly from g_{AB} and δ^B_A with appropriate symmetry and grading factors. Similarly the Pauli–Loubanski operator can be written covariantly as $W_{ABC} = M_{AB}R_C + \cdots$.

and $(t, 1)$ a translation on the space. The identity, inverse, product, and the action of $ISO(d, 2)$ on the manifold are respectively given by $(0, 1)$, $(t, \Lambda)^{-1} = (-\Lambda^{-1}t, \Lambda^{-1})$, $(t, \Lambda)(t', \Lambda') = (t + \Lambda t', \Lambda\Lambda')$ and $(t, \Lambda)x = \Lambda x + t$. Let Γ'_0 be a finite dimensional representation of $SO(d, 2)$ carried by infinitely differentiable Borel functions $\phi(x)$ for any point $x \equiv (x^a) \equiv (x^\mu, x^d, x^{d+1})$, and taking values in $\mathbb{C}$. We shall denote the carrier space by $V'_0 = \mathbb{C}^\infty(ISO(d, 2)/SO(d, 2), \mathbb{C})$. $\Phi'_0(t, \Lambda)$ will denote the operators of the representation corresponding to an element (t, Λ) of $ISO(d, 2)$, and the representation will be denoted by the pair (Φ'_0, V'_0). The covariant representation of $ISO(d, 2)$ is a representation induced from the representation Γ'_0 of $SO(d, 2)$ given by

$$\Phi'_0(t, \Lambda)\phi'_0(x) = \Gamma'_0(\Lambda)\phi'_0(\Lambda^{-1}(x - t)). \tag{6}$$

In the case of a scalar representation $\Gamma'_0(\Lambda) = I$. This representation provides as usual a representation of the algebra $iso(d, 2)$ given by

$$\Phi'_0(J_{ab})\phi'_0(x) = i\left(x_a\frac{\partial}{\partial x^b} - x_b\frac{\partial}{\partial x^a}\right)\phi'_0(x) + \Gamma'_0(J_{ab})\phi'_0(x) \tag{7}$$

$$\Phi'_0(P_a)\phi'_0(x) = i\frac{\partial}{\partial x^a}\phi'_0(x)\,. \tag{8}$$

This representation extends naturally to a representation of the universal enveloping algebra $U(iso(d, 2))$ by defining $\Phi'_0(1)\phi'_0(x) = \phi'_0(x)$, 1 being the identity of $U(iso(d, 2))$. Again for a scalar representation, $\Gamma'_0(J_{ab}) = 0$.

According to [24] and [31], the above representation is equivalent to a representation of $iso(d, 2)$ produced from the representation Γ'_0 of its subalgebra $so(d, 2)$, defined as follows. Let $U(iso(d, 2))$ be regarded as a left $U(so(d, 2))$-module. This means that the basis of $U(iso(d, 2))$ will be of the form

$$P^r = \prod P_0{}^{r_0} P_1{}^{r_1} \cdots P_d{}^{r_d} P_{d+1}{}^{r_{d+1}} \tag{9}$$

for all $r = (r_0, r_1 \ldots r_d, r_{d+1}) \in \mathbb{N}^{d+2}$, and a general element X of $U(iso(d, 2))$ is given by

$$X = \sum A_r P^r \tag{10}$$

where $A_r \in U(so(d, 2))$. Γ'_0 is carried by infinitely differentiable functions defined on $U(iso(d, 2))$ regarded as a left $U(so(d, 2))$ module, and taking values in $\mathbb{C}$. We shall denote this space of functions by $V_0 = Hom_{U(so(d,2))}(\mathcal{P}, \mathbb{C})$ where $\mathcal{P}$ is the real vector space spanned by all combinations of P^r. Then the produced algebra representations are defined for $\phi_0 \in V_0$ by

$$\Phi_0(X)\phi_0(Y) = \phi_0(YX) \qquad \text{and} \qquad \phi_0(AX) = \Gamma'_0(A)\phi_0(X) \tag{11}$$

where $X, Y \in U(iso(d, 2))$ and $A \in U(so(d, 2))$. It can also be shown [24] that this definition is equivalent to the following definition of produced representations:

$$\Phi(X)\phi_0(P) = \sum \Gamma'_0((PX)_r)\phi_0(P^r) \tag{12}$$

where $X \in iso(d, 2)$, $P \in \mathcal{P}$ and $(PX)_r \in U(so(d, 2))$ are to be interpreted as the $U(so(d, 2))$-combinations of PX in $U(iso(d, 2))$ regarded as an $U(so(d, 2))$-module (see equation (10)). Following [25, 31], for each element $\phi'_0 \in V'_0$, we define a function ϕ_0 by

$$\phi_0(X) = \Phi'_0(X)\phi'_0(x_0) \tag{13}$$

where $x_0 \in ISO(d, 2)/SO(d, 2)$ is stable under $SO(d, 2)$. Then ϕ_0 satisfies the definition (11) of the produced algebra representation, and lies in V_0. Also for a ψ'_0 such that $\psi'_0 = \Phi'_0(X)\phi'_0$ for some X of $iso(d, 2)$, then there exists a ϕ_0 of V_0 defined by (13) using ψ'_0 above, and satisfying $\psi_0 = \Phi_0(X)\phi_0$. Thus the representations (Φ'_0, V'_0) and

(Φ_0, V_0) are equivalent; in particular, $\phi_0'(x_0) = \phi_0(1)$. An explicit realization of functions $\phi_0'(x)$ expressed in terms of $\phi_0(P^r)$ that exhibits the above equivalence is given by:

$$\phi_0'(x) = \sum_{r_a} \prod_a (1/r_a!)(-ix^a)^{r_a} \phi_0(P^r)$$

where $a = 0, 1, \ldots, d, d+1$ and x^a take any real value. Then it can be shown, using the definition of the produced representation (11) that relations (6) are satisfied.

We can now proceed to construct the representation (ϕ, V) of $iosp(d, 2/2)$ produced by the trivial representation of $osp(d, 2/2)$. This is precisely what should be called a covariant scalar representation of $iosp(d, 2/2)$. The definition of the produced Lie superalgebra representations is the same as for Lie algebras mentioned above. The $U(iosp(d, 2/2))$ regarded as a $U(osp(d, 2/2))$ module has basis of the form $P^r Q^s$ with P^r as in (10) and $Q^s = Q_1^{s_1} Q_2^{s_2}$ where $s_1, s_2 \in (0, 1)$ and $s \in (0, 1) \times (0, 1)$. Let Γ be a representation of $osp(d, 2/2)$. The carrier space consists of linear functions defined on $P^r Q^s$ and thus $V = Hom_{U(iosp(d,2))}(\mathcal{P}', \mathbb{C})$ where $\mathcal{P}'$ is spanned by real combinations of the basis elements $P^r Q^s$. The produced superalgebra representation is defined by

$$\begin{aligned} \Phi(X)\phi(P^r Q^s) &= \phi(P^r Q^s X) \\ \phi(AP^r Q^s) &= \Gamma(A)\phi(P^r Q^s) \end{aligned} \tag{14}$$

where $A \in U(osp(d, 2/2))$, $X \in iosp(d, 2/2)$ and $\phi \in V$. For the covariant scalar representation the $osp(d, 2/2)$ is represented trivially and thus Γ =0 (= Γ' when restricted to $so(d, 2)$). Note now that every $\phi \in V$, when defined on $\mathcal{P} \subset \mathcal{P}'$, is a member of V_0, and via the equivalence mentioned above gives a member of V_0'. Moreover, from the definition of produced superalgebra representation, there is a one to one equivalence between a $\phi \in V$ and a set of four functions defined solely on P^r, namely $\phi(P^r)$, $(\Phi(Q_\alpha)\phi)$, $(\Phi(Q_\alpha Q_\beta)\phi)$. Thus, using this we regard an element of V as comprising the following set of four functions defined on $ISO(d, 2)/SO(d, 2)$:

$$\phi(x) \qquad (\Phi(Q_\alpha)\phi)(x) = \phi(x, \alpha) \qquad (\Phi(Q_1 Q_2)\phi)(x) = \phi(x, 12). \tag{15}$$

Finally, the action of the operators $\Phi(X)$ for every $X \in iosp(d, 2/2)$ can be evaluated by calculating $\Phi(X)$ on these four functions, using relations (14), (11), (7)–(8), the above realization of $\phi_0(x)$ and the commutation relations (2) (the dashes have been dropped out via the equivalence mentioned above). This action for the covariant $iosp(d, 2/2)$ scalar multiplet is given by

$$\Phi(J_{ab})\phi(x) = \Phi_0(J_{ab})\phi(x) \qquad \Phi(P_a)\phi(x) = \Phi_0(P_a)\phi(x) \tag{16a}$$

$$\Phi(J_{ab})\phi(x, \alpha) = \Phi_0(J_{ab})\phi(x, \alpha) \qquad \Phi(P_a)\phi(x, \alpha) = \Phi_0(P_a)\phi(x, \alpha) \quad \alpha = 1, 2 \tag{16b}$$

$$\Phi(J_{ab})\phi(x, \alpha\beta) = \Phi_0(J_{ab})\phi(x, \alpha\beta) \qquad \Phi(P_a)\phi(x, \alpha\beta) = \Phi_0(P_a)\phi(x, \alpha\beta) \quad \alpha, \beta = 1, 2 \tag{16c}$$

$$\Phi(K_{\alpha\beta})\phi(x) = 0 \qquad \Phi(Q_\alpha)\phi(x) = \phi(x, \alpha) \tag{16d}$$

$$\Phi(K_{\alpha\beta})\phi(x, \gamma) = \epsilon_{\alpha\gamma}\phi(x, \beta) + \epsilon_{\beta\gamma}\phi(x, \alpha) \qquad \Phi(Q_\alpha)\phi(x, \beta) = -\phi(x, \alpha\beta) \tag{16e}$$

$$\Phi(K_{\alpha\beta})\phi(x, \beta\gamma) = 0 \qquad \Phi(Q_\alpha)\phi(x, \beta\gamma) = 0 \tag{16f}$$

$$\Phi(L_{a\alpha})\phi(x) = g_{ab}x^b\phi(x, \alpha) \tag{16g}$$

$$\Phi(L_{a\alpha})\phi(x, \beta) = -g_{ab}x^b\phi(x, \alpha\beta) - i\epsilon_{\alpha\beta}\Phi_0(P_a)\phi(x) \tag{16h}$$

$$\Phi(L_{a\alpha})\phi(x, \beta\gamma) = -i\epsilon_{\beta\gamma}\Phi_0(P_a)\phi(x, \alpha)\,. \tag{16i}$$

An indefinite inner product is given by [24, 25]

$$(\phi, \psi) = \int \mathrm{d}^{d+2}x \; \varepsilon^{\alpha\beta}[\phi^*(x, \alpha\beta)\psi(x) - \phi^*(x)\psi(x, \alpha\beta) - \phi^*(x, \alpha)\psi(x, \beta) \\ +\phi^*(x, \beta)\psi(x, \alpha)]\,. \tag{17}$$

Under this inner product, for functions with appropriate boundary conditions, the $iso(d, 2)$ and $sp(2, \mathbb{R})$ generators are represented by Hermitian operators while the rest are anti-Hermitian.

Irreducibility of the covariant scalar multiplet demands that each of the Casimir operators has the same eigenvalue on all the four functions above. In particular we demand that

$$\begin{aligned} C_2\phi(x) &= \Phi(P_aP^a)\phi(x) + \Phi(Q_\alpha Q^\alpha)\phi(x) \\ &= \Phi(P_aP^a)\phi(x) + 2i\phi(x, 12) = \lambda\phi(x) \\ C_2\phi(x, \alpha) &= \Phi(P_aP^a)\phi(x, \alpha) = \lambda\phi(x, \alpha) \\ C_2\phi(x, \alpha\beta) &= \Phi(P_aP^a)\phi(x, \alpha\beta) = \lambda\phi(x, \alpha\beta) \end{aligned} \tag{18}$$

where λ is the constant eigenvalue of C_2 characterizing the irreducible $iosp(d, 2/2)$ multiplet. Finally we can introduce an (anti-Hermitian) ghost number operator given by

$$Q_c = i/2(K_{11} - K_{22}) \tag{19}$$

so that the functions of definite ghost number are: $\phi(x)$ and $\phi(x, 12)$ with ghost number 0, and $\phi(x, 1) \pm \phi(x, 2)$ with ∓ 1, respectively.

3. BFV-BRST quantization of the scalar relativistic particle

As is well known [28, 32] the BFV canonical quantization of constrained Hamiltonian systems [13] uses an extended phase-space description in which, to each first-class constraint, a pair of conjugate 'ghost' variables (of Grassmann parity opposite to that of the constraint) is introduced. Here we follow this procedure for the scalar relativistic particle. Although our notation is adapted to the massive case, $m > 0$, as would follow from the second order action corresponding to extremization of the proper length of the particle world line, an analysis of the *fundamental* Hamiltonian description of the first-order action [28] leads to an equivalent picture (with an additional mass parameter $\mu \neq 0$ supplanting m in appropriate equations, and permitting $m \to 0$ as a smooth limit). In either case, for the scalar particle the primary first-class constraint is the mass-shell condition $(P^2 - m^2)$, where $P^2 = P_\mu P^\mu$; including the corresponding Lagrange multiplier λ as an additional dynamical variable then leads to a secondary constraint, reflecting conservation of its conjugate momentum. The quantum formulation, to which we proceed directly, should be consistent with the equations of motion and gauge fixing at the classical level. We choose below to work in the class [14, 33] $\dot{\lambda} = 0$; moreover, with the restriction to *orientation preserving* diffeomorphisms (world-line reparametrizations), it is sufficient to choose $\lambda > 0$ (a parallel treatment applies for $\lambda < 0$). This restriction will also be essential in establishing the equivalence to the algebraic approach of section 2 above.

State space and wavefunctions

The BFV extended phase space [28] for the BRST quantization of the scalar relativistic particle is taken to comprise the following canonical variables:

$$x^\mu(\tau),\, p_\mu(\tau), \quad \omega(\tau), \pi(\tau), \quad \eta^\alpha(\tau), \rho_\alpha(\tau) \tag{20}$$

where ω parametrizes the Lagrange multiplier $\lambda = e^{\omega}$, π is the momentum conjugate to ω, and η^{α}, ρ_{α}, $\alpha = 1, 2$ are the Grassmann odd BFV extended phase space variables. The operators corresponding to the above set satisfy the following commutation relations†:

$$[X_{\mu}, P_{\nu}] = -i g_{\mu\nu} \tag{21a}$$

$$[\hat{\omega}, \hat{\pi}] = i \tag{21b}$$

$$[\eta^{\alpha}, \rho_{\beta}] = -i\delta^{\alpha}_{\beta} . \tag{21c}$$

The ghost number operator Q_c is defined by

$$Q_c = (i/2)(\eta^{\alpha}\rho_{\alpha} - \rho_{\alpha}\eta^{\alpha}). \tag{22}$$

The canonical BRST operator is given by

$$\Omega = \eta^1\hat{\pi} + \eta^2(P^2 - m^2) . \tag{23}$$

We shall also use the corresponding anti-BRST operator

$$\bar{\Omega} = \frac{i}{2}(\rho_2\hat{\pi} - \rho_1(P^2 - m^2)). \tag{24}$$

The gauge-fixing operator [13] Ψ which will lead to the appropriate effective Hamiltonian is given by

$$\begin{aligned} \Psi &= -\tfrac{1}{2}e^{\hat{\omega}}\rho_2 \\ H &= i[\Psi, \Omega] = -\tfrac{1}{2}(e^{\hat{\omega}}\eta^1\rho_2 + e^{\hat{\omega}}(P^2 - m^2)). \end{aligned} \tag{25}$$

Consider the linear representation of the algebra of (21*a*), (21*b*) on say coordinate space:

$$\begin{aligned} &X^{\mu}|x^{\mu}\rangle = x^{\mu}|x^{\mu}\rangle && P_{\mu}|x^{\mu}\rangle = -i\frac{\partial}{\partial x^{\mu}}|x^{\mu}\rangle \\ &\langle x^{\mu\prime}|x^{\mu}\rangle = \delta(x^{\mu\prime} - x^{\mu}) && \langle x^{\mu}|p_{\mu}\rangle = \frac{1}{(2\pi)^{d/2}}e^{-ix^{\mu}p_{\mu}} \\ &\hat{\omega}|\omega\rangle = \omega|\omega\rangle && \hat{\pi}|\omega\rangle = i\frac{\partial}{\partial\omega}|\omega\rangle \\ &\langle\omega'|\omega\rangle = \delta(\omega' - \omega) && \langle\omega|\pi\rangle = \frac{1}{(2\pi)^{1/2}}e^{i\omega\pi} . \end{aligned} \tag{26}$$

We also recognize (21*c*) as a b, c algebra [28], where b stands for $i\rho_{\alpha}$ and c for η^{α}. Then the algebra admits a representation on a four dimensional linear space with basis denoted by $|\pm\pm\rangle, |\pm\mp\rangle$, and the action of η^{α} and $i\rho_{\alpha}$, is given by

$$\begin{aligned} &\eta^1|--\rangle = |+-\rangle && \eta^1|-+\rangle = |++\rangle \\ &\eta^2|--\rangle = |-+\rangle && \eta^2|+-\rangle = -|++\rangle \\ &i\rho_1|+-\rangle = |--\rangle && i\rho_1|++\rangle = |-+\rangle \\ &i\rho_2|-+\rangle = |--\rangle && i\rho_2|++\rangle = -|+-\rangle . \end{aligned} \tag{27}$$

The non-zero inner products between these states are given by:

$$\langle--|++\rangle = -\langle++|--\rangle = i \qquad \langle-+|+-\rangle = -\langle+-|-+\rangle = -i . \tag{28}$$

† The choice of ω and π as conjugate variables corresponds to a choice of a particular inner product, and hence Hermitian canonical conjugate to λ, for the direct problem of quantization on the half-line $\lambda > 0$. The ultimate determinant of these choices is the identification with the produced representation of section 2 above (see below).

The above representations and inner products imply the Hermiticity conditions

$$X^\dagger_\mu = X_\mu \qquad P^\dagger_\mu = P_\mu \qquad \hat{\omega}^\dagger = \hat{\omega} \qquad \hat{\pi}^\dagger = \hat{\pi} \tag{29}$$

$$(\eta^\alpha)^\dagger = \eta^\alpha \qquad (\rho_\alpha)^\dagger = -(\rho_\alpha)\,. \tag{30}$$

Finally, with the identity operator given by

$$I = \sum\nolimits_{\sigma\sigma'=\pm}\int_\infty \mathrm{d}^d x\ \mathrm{d}\omega\ (-i)(-1)^{(1-\sigma')/2}|x^\mu, \omega, \sigma, \sigma'\rangle\langle x^\mu, \omega, -\sigma, -\sigma'| \tag{31}$$

a general state $|\psi\rangle$ of the system is

$$\begin{aligned} |\psi\rangle &= \sum\nolimits_{\sigma\sigma'=\pm}\int_\infty \mathrm{d}^d x\ \mathrm{d}\omega\ |x^\mu, \omega, \sigma, \sigma'\rangle\psi_{\sigma\sigma'}(x^\mu, \omega, \tau) \\ \psi_{\sigma\sigma'}(x^\mu, \omega, \tau) &= -i(-1)^{(1-\sigma')/2}\langle x^a, \omega, -\sigma, -\sigma'|\psi\rangle\,. \end{aligned} \tag{32}$$

The inner product $\langle\phi|\psi\rangle$ in terms of wavefunctions is given by

$$\langle\phi|\psi\rangle = (-i)\int_\infty \mathrm{d}^d x\ \mathrm{d}\omega\ (-1)^{(1-\sigma')/2}\sum\nolimits_{\sigma\sigma'=\pm}\phi*_{\sigma\sigma'}(x^\mu, \omega, \tau)\psi_{-\sigma-\sigma'}(x^\mu, \omega, \tau)\,. \tag{33}$$

As usual the wavefunctions ψ are required to vanish at $\omega = \pm\infty$. With respect to the previously defined ghost number operator, (22), the kets $|\pm\pm\rangle$, $|\pm\mp\rangle$, and corresponding wavefunction components $\psi_{\pm\pm}$, $\psi_{\pm\mp}$, have eigenvalues $\pm 1, 0$, respectively.

The gauge-invariant physical states can now be identified [28] by imposing the Schrödinger equation $i\mathrm{d}|\psi\rangle/\mathrm{d}\tau = H|\psi\rangle \equiv i[\Psi, \Omega]|\psi\rangle$ and computing the cohomology of the BRST operator Ω. However, in order to exhibit the $iosp(d, 2/2)$ symmetry in the above quantization procedure at the level of state space, it is convenient to use equivalent BRST and gauge-fixing operators Ω', Ψ', which can be more directly expressed in terms of the superalgebra generators. The wavefunctions $\psi_{\sigma\sigma'}(x^\mu, \omega, \tau)$ can then be readily identified with those of the functions of section 2 above which carry the $iosp(d, 2/2)$ produced representation (with appropriate boundary conditions). Our final identification of physical states will then follow with respect to the cohomology of the transformed BRST operator.

Consider the following canonical transformation on the classical dynamical variables of the extended phase space [16]:

$$\begin{aligned} i\rho'_\alpha &= e^{-\omega} i\rho_\alpha \\ \eta'^\alpha &= e^\omega \eta^\alpha \\ \hat{\pi}' &= \hat{\pi} - (\eta^2\rho_2 - \rho_1\eta^1) \end{aligned} \tag{34}$$

with the remainder invariant. At the quantum level the corresponding BRST and anti-BRST operators $\Omega' = \eta'^1\hat{\pi}' + \eta'^2(P^2 - m^2)$, $\bar{\Omega}' = \frac{i}{2}(\rho'_2\hat{\pi}' - \rho'_1(P^2 - m^2))$ can be written as

$$\begin{aligned} \Omega' &= \eta^1 : e^{\hat{\omega}}\hat{\pi} : + e^{\hat{\omega}}\eta^2(P^2 - m^2) - e^{\hat{\omega}}\eta^2\rho_2\eta^1 \\ \bar{\Omega}' &= (i/2)(\rho_2 : e^{\hat{\omega}}\hat{\pi} : - e^{\hat{\omega}}\rho_1(P^2 - m^2) + e^{\hat{\omega}}\rho_1\rho_2\eta^1) \end{aligned} \tag{35}$$

where the symmetric ordering

$$: e^{\hat{\omega}}\hat{\pi} := (1/2)(e^{\hat{\omega}}\hat{\pi} + \hat{\pi}e^{\hat{\omega}}) = \hat{\pi}e^{\hat{\omega}} + (i/2)e^{\hat{\omega}} \tag{36}$$

has been introduced. It is also convenient to define [16] X_α and Q_α ($\alpha = 1, 2$)by

$$\begin{aligned} Q_1 &= (i/2\sqrt{2})(2\eta^1 + i\rho_2) \qquad & Q_2 &= (i/2\sqrt{2})(2\eta^1 - i\rho_2) \\ X_1 &= (i/\sqrt{2})(i\rho_1 - 2\eta^2) \qquad & X_2 &= (i/\sqrt{2})(-i\rho_1 - 2\eta^2) \\ [Q_\alpha, X_\beta] &= -i\epsilon_{\alpha\beta}. \end{aligned} \tag{37}$$

In terms of these variables we attain the following simple forms for the BRST, gauge-fixing and Hamiltonian operators:

$$\begin{aligned}
\Omega' &= (-i/\sqrt{2})(:\hat{\pi}e^{\hat{\omega}}:(Q_1+Q_2)+(X_1+X_2)H) \\
\bar{\Omega}' &= (-i/\sqrt{2})(:\hat{\pi}e^{\hat{\omega}}:(Q_1-Q_2)+(X_1-X_2)H) \\
\Psi' &= -(1/2)\rho_2 = (1/\sqrt{2})(Q_1-Q_2) \\
H' &= i[\Psi',\Omega'] = -(1/2)e^{\hat{\omega}}((P^2-m^2)+2iQ_1Q_2) \equiv H.
\end{aligned} \tag{38}$$

Realization of the $iosp(d,2/2)$ superalgebra

The realization of $iosp(d,2/2)$ provided by the extended BFV-BRST quantization as described above is formulated in terms of the operators X^μ, P_μ together with Q_α, X_α and $X_+ = \tau I$, $P_- = H$, $P_+ = e^{-\hat{\omega}}$, $X_- =: \hat{\pi}e^{\hat{\omega}}:$. With the non-zero commutation relations between these variables being

$$[X_\mu, P_\nu] = -ig_{\mu\nu} \qquad [X_-, P_+] = i$$

$$[X_-, P_-] = -iP_+^{-1}P_- \qquad [X_\alpha, P_-] = iP_+^{-1}Q_\alpha \qquad [X_\mu, P_-] = iP_+^{-1}P_\mu$$

it can be checked that the following generators do indeed satisfy the commutation relations of $osp(d,2/2)$:

$$\begin{aligned}
&J_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu \qquad J_{+-} = X_-P_+ + X_+P_- \qquad J_{\mp\mu} = \mp X_\mp P_\mu - X_\mu P_\mp \\
&K_{\alpha\beta} = -i(X_\alpha Q_\beta + X_\beta Q_\alpha) \qquad L_{\mu\alpha} = X_\mu Q_\alpha - X_\alpha P_\mu \qquad L_{\mp\alpha} = \mp X_\mp Q_\alpha - X_\alpha P_\mp
\end{aligned} \tag{39}$$

where $L_{-1} = -i/\sqrt{2}(\Omega' + \bar{\Omega}')$, $L_{-2} = -i/\sqrt{2}(\Omega' - \bar{\Omega}')$. Together with P_μ, $P_\pm$, Q_α, these generators close† on the inhomogeneous form $iosp(d,2/2)$ (see equation (2) above). It is clear that the $(d+2)$-dimensional coordinates x_μ, $x_\pm$, x_α and momenta P_μ, $P_\mp$, Q_α are *not* all canonically conjugate. In particular X_+, proportional to the identity operator, simply re-scales kets (at time τ) by τ, while P_- is identified with the Hamiltonian, a function of the other variables (whose action also sets the rate of time development of kets via the Schrödinger equation).

The final stage in the analysis is the identification of the (τ-dependent) wavefunctions $\psi_{\sigma\sigma'}$ with the functions over x^a which carry the produced representation in section 2 above. To facilitate this comparison we introduce kets and wavefunctions dependent on $p_+ = e^{-\omega}$ by a change of variables. As $e^{-\omega}$ is a monotonic differentiable function, $|x^\mu, p_+, \sigma, \sigma'\rangle$ can be defined by

$$|x^\mu, \omega, \sigma, \sigma'\rangle = p_+^{1/2}|x^\mu, p_+, \sigma, \sigma'\rangle \qquad \langle p_+|p'_+\rangle = \delta(p_+ - p'_+). \tag{40}$$

Then the completeness relation becomes

$$I = \sum_{\sigma\sigma'=\pm}\int \mathrm{d}^dx \int_0^\infty \mathrm{d}p_+\, (-i)(-1)^{(1-\sigma')/2}|x^\mu, p_+, \sigma, \sigma'\rangle\langle x^\mu, p_+, -\sigma, -\sigma'| \tag{41}$$

while

$$\begin{aligned}
\psi_{\sigma\sigma'}(x^\mu, p_+, \tau) &= -i(-1)^{(1-\sigma')/2}\langle x^\mu, p_+, -\sigma, -\sigma'|\psi(\tau)\rangle \\
&= -i(-1)^{(1-\sigma')/2}e^{\omega/2}\psi_{\sigma\sigma'}(x^a, \omega, \tau)\,.
\end{aligned} \tag{42}$$

† In covariant notation (see the footnote to (2)) this realization can be written simply in terms of $X_AR_B - (-1)^{|AB|}X_BR_A$ and R_A. However, as noted above, the X_A and R_B are *not* canonically conjugate.

It should be noted that the domain of p_+ is restricted to be $p_+ \in (0, \infty)$ as $\omega \in (-\infty, \infty)$ and this will result in wavefunctions $\psi_{\sigma\sigma'}(x^\mu, p_+, \tau)$ which vanish when p_+ approaches zero or infinity. In the p_+ representation the operator X_- is realized† as $i\partial/\partial p_+$, while the inner product becomes

$$\langle\phi|\psi\rangle = -i\int \mathrm{d}^d x \int_0^\infty \mathrm{d}p_+ \sum\nolimits_{\sigma\sigma'=\pm} (-1)^{(1+\sigma')/2} \phi*_{\sigma\sigma'}(x^\mu, p_+, \tau)\psi_{-\sigma-\sigma'}(x^\mu, p_+, \tau). \tag{43}$$

The action of the operators (39) on the wavefunctions $\psi_{\sigma\sigma'}(x^a, p_+, \tau)$ is given in the appendix together with the Schrödinger equation for them ((see equations (A1)–(A4)). It can be easily seen that with the identifications

$$\begin{aligned}
&\phi(x^a, p_+, x_+) = \psi_{+-}\\
&\phi(x^a, p_+, x_+, 1) = (i/\sqrt{2})(\psi_{--} - (1/2)\psi_{++})\\
&\phi(x^a, p_+, x_+, 2) = (i/\sqrt{2})(\psi_{--} + (1/2)\psi_{++})\\
&\phi(x^a, p_+, x_+, 12) = (1/2)\psi_{-+}
\end{aligned} \tag{44}$$

the representation obtained in the appendix (see equation (A2)) is identical with that constructed in the produced representation in section 2 above, provided that the Fourier transforms of the functions on x_- have support on $p_+ \in (0, \infty)$ in conformity with the present construction.

Having established for this model the equivalence of the physical quantization construction with the algebraic produced representation, we can now proceed to identify physical states (in either picture) by computing the BRST cohomology. The BRST-invariant states are defined by the condition $\Omega'|\phi\rangle = 0$, with general solution $|\phi\rangle = |\psi\rangle + \Omega'|\chi\rangle$ and $|\psi\rangle$ not in the range of Ω'. Then using the information of the appendix, the above condition gives the following restrictions for the wavefunctions $\psi_{\sigma\sigma'}(x^a, p_+, \tau)$:

$$i\frac{\mathrm{d}}{\mathrm{d}p_+}\psi_{--} = 0 \tag{45a}$$

$$H\psi_{--} = i\frac{\mathrm{d}}{\mathrm{d}\tau}\psi_{--} = 0 \tag{45b}$$

$$\left(i\frac{\mathrm{d}}{\mathrm{d}p_+}\right)\left(\frac{1}{2}\right)\psi_{-+} + i\frac{\mathrm{d}}{\mathrm{d}\tau}\psi_{+-} = 0 \tag{45c}$$

where the Schrödinger equation has been used for the last two expressions. At the algebraic level the above restrictions arise by demanding the vanishing of $(L_{-1}+L_{-2})\psi_{+-}$, $(L_{-1}+L_{-2})\psi_{-+}$, and $(L_{-1}+L_{-2})\psi_{++}$, leading to (45*a*)–(45*c*), respectively, while the condition $(L_{-1}+L_{-2})\psi_{--} = 0$ is identically satisfied. Thus the wavefunction ψ_{--} of ghost number -1 is BRST-invariant, by (45*a*), (45*b*) is independent of $\tau = x_+$ and p_+, and by (45*b*) and (A4) satisfies the Klein-Gordon equation. In conclusion, we see ψ_{--} and any BRST-equivalent states of ghost number -1 are in direct correspondence with the physical states.

These results have equivalents at the level of the produced algebra representation via (44). The BRST invariance conditions become conditions for the vanishing of $(L_{-1}+L_{-2})$. Transforming from x_- to p_+, and using the irreduciblity requirement for the multiplet (with $\lambda = m^2$ in (18)), we see that the vanishing of $(L_{-1}+L_{-2})$ on $\phi(x)$ will lead to

† The wavefunctions in the x_- representation are given by $\psi_{\sigma\sigma'}(x^a, x_-, x_+) = \int \mathrm{d}p_+\, e^{-ix_-p_+}\psi_{\sigma\sigma'}(x^a, p_+, x_+)$.

(45a), on $\phi(x,\alpha)$ will both lead to (45c), and on $\phi(x,12)$ will lead to (45b). Again, the physical states are identified with $(-i/2)\sqrt{2}(\phi(x^{\mu},0,0,1)+\phi(x^{\mu},0,0,2))$. Finally note that the requirement that the $iosp(d,2/2)$ states should satisfy the Schrödinger equation is identical with the demand that the covariant massive scalar $iosp(d,2/2)$ multiplet should be irreducible. That is, relation (18) should be satisfied, and we easily see that the effective Hamiltonian should have the form $P_{-}=H=-\frac{1}{2}P_{+}^{-1}(P_{\mu}P^{\mu}-m^{2}+Q_{\alpha}Q^{\alpha})$. Finally it should be mentioned that the identification of the BRST operator with the superalgebra element $(L_{-1}+L_{-2})$ is merely a consequence of the realization (39) and the analysis of [16, 17] and does not constitute a unique choice of this operator at the superalgebra level. In fact one could argue that one could describe BFV systems possessing an $iosp(d,2/2)$ symmetry in a way that the BRST-operator appears as some admissible linear combination of odd nilpotent elements that the superalgebra possesses.

4. Conclusions

In this paper we have considered in detail the canonical BFV-BRST quantization of the scalar relativistic particle and its relationship to the extended quantization supersymmetry superalgebra $iosp(d,2/2)$. In particular, a certain type of covariant scalar produced module of the latter is identified with the extended state space of the particle quantization in the usual wavefunction and b,c algebra constructions.

The features of our approach have been the consistent treatment of the quantization problem for the Lagrange multiplier on the half-line ($p_{+}\equiv\lambda^{-1}>0$ in our notation) which is necessary for the identification of the $iosp(d,2/2)$ covariance (see comments below). Although the emergence of the extended $iosp(d,2/2)$ algebra may seem fortuitous in this particle quantization example (section 3), the equivalence with the canonical produced algebra construction (section 2) suggests that the phenomenon is quite universal. Thus it might be expected that the BFV-BRST quantization using a broad class of gauge-fixing fermions corresponding to admissible gauge fixings (see below) of the general type [28] $\dot{\lambda}=F(\lambda)$ would also admit the extended supersymmetry. With regard to the identification with the produced representation, it must be noted that the natural inner product (section 2) is supplanted by a pointwise inner product (section 3) which in principle is proper-time dependent. Of course, for states obeying Schrödinger's equation, this inner product is necessarily proper time *independent*.

The $iosp(d,2/2)$ representation (section 3) has been explicitly shown to be built in terms of only $d+1$ canonically conjugate pairs of bosonic variables (together with the extended fermionic modes), with one momentum component, p_{-}, identified with the Hamiltonian H, and its 'conjugate' variable x_{+} set equal to the proper time τ. At the $(d,2/2)$-dimensional level the realization is analogous to a reduced phase space or Hamiltonian reduction approach, with constraints solved explicitly in terms of an independent set of variables; related constructions have also been proposed abstractly for 'covariant' quantization algebras [18].

In the present work no direct appeal is made to superfield constructions. Although in this case the representation found can in fact be shown to be identical to a superfield version [17], our approach is more general and is still possible for cases where superfield considerations are inappropriate or not available. Indeed, the general theory of produced representations as exemplified here, provides [24, 25] a formal link between abstract representation theory and more heuristic superfield methods.

Since the $iosp(d,2/2)$ covariance is established at the level of the state space, we have not entered into considerations of the path integral representation of the canonical action and generating function [15, 16]. Nevertheless, for the present case the evolution kernel can in principle be evaluated directly. The derived causal scalar particle Green function would then establish the connection with the second-quantized theory. The choice $p_+ \equiv \lambda^{-1} > 0$ corresponds, with the gauge class used, to an admissible section [28] of the space of gauge orbits, including the global modular transformation (in this case an orientation-reversing diffeomorphism, which together with the identity forms a $\mathbb{Z}_2$ group). The quantization is thus carried out for the unoriented scalar particle; the opposite sign would correspond to the unoriented scalar antiparticle [28], and indeed the usual extension whereby $\phi(-p_+) \sim \phi^*(p_+)$ is consistent with this PCT transformation [34].

It is a striking fact that both for the massless and massive particle, the extended quantization symmetry involves *massless* representations at the $(d, 2/2)$ level, since the identification of the (inverse) Lagrange multiplier with p_+, and of the Hamiltonian with p_- is perfect for the interpretation (on physical states) of the Schrödinger equation $H = -(1/2)p_+^{-1}(p_\mu p^\mu + Q_\alpha Q^\alpha - m^2)$ as the vanishing of the quadratic Casimir, in light-cone coordinates for the 2 extra bosonic directions. The 'dimensional reduction' from $(d, 2/2)$ to d dimensions appears here in the analysis of physical states directly via the wavefunctions' independence of $x_\pm$, rather than through a Parisi-Sourlas [4] cancellation mechanism, although this has been established abstractly for Greens functions in the case of irreducible *momentum* representations of $iosp(d, 2/2)$ by Cornwell and Hartley [24, 25]. Similar reductions have been discussed in the context of loop integrals in quantum field theory [35, 36, 10].

Future work [3] in the programme initiated here will extend the algebraic analysis to other first quantized systems such as the spinning particle and superparticle, as well as to gauge field theories such as Yang–Mills–Shaw. General questions will be to confirm the covariance of the canonical approach and ghost systems [37] with respect to an extended orthosymplectic spacetime symmetry, particularly with regard to issues of modular invariance and the relation of Teichmüller space to the appropriate induced or produced representation theory. At this level should also emerge the reasons for the use in the literature of $(d/2)$- as opposed to $(d, 2/2)$-dimensional superfield formalisms for covariant quantization and discussions of renormalization [8, 9], and the connection with geometrical approaches based on coset space dimensional reduction [38]. Finally, the algebraic structure of quantization using BRST symmetry is extremely rich and flexible, as has been demonstrated by investigations of alternative schemes in the context of internal symmetry [39] and of cohomological approaches [40]. It can be expected that the study of extended quantization symmetries along the lines advocated here may lead to consistent ways of implementing covariant quantization in systems such as string field theories where the gauge algebra presents technical difficulties. In any case, it is reasonable to assert that a 'Wigner' type classification of admissible 'gauge multiplets' may evolve from this viewpoint.

Acknowledgments

The authors are grateful to A J Bracken, R Delbourgo, D Kreimer, D McAnally and J Govaerts for helpful hints and comments which facilitated the development of the present work. This work was supported under grant A69332249 from the Australian Research Council.

Appendix

The action of the transformed operators $X_\alpha, Q_\alpha, \hat{\pi}$ and X_- on the fundamental kets $|x^\mu, p_+, \sigma\sigma'\rangle$ can easily be computed using (26), (27), (36) and (37). The action of any element A, of $iosp(d, 2/2)$ or of an operator corresponding to phase space variables, on the functions $\psi_{\sigma\sigma'}(x^\mu, p_+, \tau)$ is given by

$$A\psi_{\sigma\sigma'}(x^\mu, p_+, \tau) = -i(-1)^{(1-\sigma')/2}\langle x^\mu, p_+, -\sigma, -\sigma'|A|\psi\rangle\,. \tag{A1}$$

On the basis of the above information we can now calculate the action of the operators (39) on $\psi_{\sigma\sigma'}(x^\mu, p_+, \tau)$:

$$\begin{aligned}
&J_{\mu\nu}\psi_{\sigma\sigma'} = i\left(x_\mu\frac{\mathrm{d}}{\mathrm{d}x^\nu} - x_\nu\frac{\mathrm{d}}{\mathrm{d}x^\mu}\right)\psi_{\sigma\sigma'} \qquad P_a\psi_{\sigma\sigma'} = i\frac{\mathrm{d}}{\mathrm{d}x^a}\psi_{\sigma\sigma'}\\
&K_{11}\psi_{--} = \tfrac{1}{2}K_{11}\psi_{++} = i\psi_{--} - (i/2)\psi_{++}\\
&K_{22}\psi_{--} = -\tfrac{1}{2}K_{22}\psi_{++} = -i\psi_{--} - (i/2)\psi_{++}\\
&K_{12}\psi_{--} = (i/2)\psi_{++} \qquad K_{12}\psi_{++} = 2i\psi_{--}\\
&K_{\alpha\beta}\psi_{\pm\mp} = 0\\
&Q_\alpha\psi_{+-} = (-1)^\alpha\frac{i}{2\sqrt{2}}\psi_{++} + \frac{i}{\sqrt{2}}\psi_{--}\\
&Q_1\psi_{-+} = Q_2\psi_{-+} = Q_1Q_2\psi_{--} = Q_1Q_2\psi_{++} = Q_1Q_2\psi_{-+} = 0\\
&Q_\alpha\psi_{++} = 2(-1)^{\alpha-1}Q_\alpha\psi_{--} = \frac{i}{\sqrt{2}}\psi_{-+}\\
&Q_1Q_2\psi_{+-} = \tfrac{1}{2}\psi_{-+}\\
&L_{-1}\psi_{--} = -L_{-2}\psi_{--} = \left(\frac{-i}{2\sqrt{2}}\right)i\frac{\mathrm{d}}{\mathrm{d}p_+}\psi_{-+} - \frac{i}{\sqrt{2}}H\psi_{+-}\\
&L_{-1}\psi_{+-} = -i\frac{\mathrm{d}}{\mathrm{d}p_+}(Q_1\psi_{+-}) \qquad L_{-2}\psi_{+-} = -i\frac{\mathrm{d}}{\mathrm{d}p_+}(Q_2\psi_{+-})\\
&L_{-1}\psi_{-+} = 2H(Q_1\psi_{+-}) \qquad L_{-2}\psi_{-+} = 2H(Q_2\psi_{+-})\\
&L_{-1}\psi_{++} = L_{-2}\psi_{++} = \left(\frac{-i}{\sqrt{2}}\right)i\frac{\mathrm{d}}{\mathrm{d}p_+}\psi_{-+} - 2\frac{i}{\sqrt{2}}H\psi_{+-}\\
&L_{\mu\alpha}\psi_{\sigma\sigma'} = -x_\mu Q_\alpha\psi_{\sigma\sigma'} + i\frac{\mathrm{d}}{\mathrm{d}x^\mu}X_\alpha\psi_{\sigma\sigma'}\\
&J_{-\mu}\psi_{\sigma\sigma'} = i\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(i\frac{\mathrm{d}}{\mathrm{d}p_+}\right)\psi_{\sigma\sigma'} + x_\mu H\psi_{\sigma\sigma'}\\
&J_{+-}\psi_{\sigma\sigma'} = x_+H\psi_{\sigma\sigma'} + i\frac{\mathrm{d}}{\mathrm{d}p_+}(p_+\psi'_{\sigma\sigma'})\,.
\end{aligned} \tag{A2}$$

From the Schrödinger equation

$$i\frac{\mathrm{d}}{\mathrm{d}\tau}|\psi_{\sigma\sigma'}\rangle = H|\psi_{\sigma\sigma'}\rangle \tag{A3}$$

and (A1) we also have

$$i\frac{\mathrm{d}}{\mathrm{d}\tau}\psi_{--} = H\psi_{--} = -\frac{1}{2}p_+^{-1}(P^2-m^2)\psi_{--}$$
$$i\frac{\mathrm{d}}{\mathrm{d}\tau}\psi_{++} = H\psi_{++} = -\frac{1}{2}p_+^{-1}(P^2-m^2)\psi_{++}$$
$$i\frac{\mathrm{d}}{\mathrm{d}\tau}\psi_{+-} = H\psi_{+-} = -\frac{1}{2}p_+^{-1}(P^2-m^2)\psi_{+-} + ip_+^{-1}Q_1Q_2\psi_{+-} \tag{A4}$$
$$i\frac{\mathrm{d}}{\mathrm{d}\tau}\psi_{-+} = H\psi_{-+} = -\frac{1}{2}p_+^{-1}(P^2-m^2)\psi_{-+}\,.$$

References

[1] Dirac P A M 1964 *Lectures on Quantum Mechanics* (new York: Belfer Graduate School of Science, Yeshiva University); 1966 *Lectures on Quantum Field Theory* (New York: Belfer Graduate School of Science, Yeshiva University)
[2] Becchi C, Rouet A and Stora R 1974 *Phys. Lett.* **52B** 344; 1976 *Ann. Phys.* **98** 287
Tyutin I V 1975 *Lebedev preprint* FIAN 39, unpublished
[3] I Tsohantjis and P D Jarvis 1996 in preparation
[4] Parisi G and Sourlas N 1979 *Phys. Rev. Lett.* **43** 744
[5] Dondi P H and Jarvis P D 1979 *Phys. Lett.* **84B** 75 (Erratum **87B** 403)
[6] Ojima I 1980 *Prog. Theor. Phys.* **64** 625
[7] Bonora L and Tonin M 1981 *Phys. Lett.* **98B** 48
[8] Delbourgo R and Jarvis P D 1982 *J. Phys. A: Math. Gen.* **15** 611
Delbourgo R, Thompson G and Jarvis P D 1982 *Phys. Rev.* D **28** 2122
[9] Joglekar S D 1991 *Phys. Rev.* D **43** 1307
[10] Ishizuka W and Kikuchi Y 1986 *Phys. Lett.* **171B** 408
[11] Berezin F A and Marinov M S 1975 *JETP Lett.* **21** 320; 1977 *Ann. Phys., NY* **104** 336
Casalbuoni R 1976 *Nuovo Cimento* A **33** 115
Barducci A, Casalbuoni R and Lusanna L 1976 *Nuovo Cimento* A **35** 377
[12] Gitman D M and Tyutin I V 1990 *Class. Quantum Grav.* **7** 2131; 1990 *Quantization of Fields with Constraints* (Berlin: Springer)
[13] Fradkin E S and Vilkovisky G A 1975 *Phys. Lett.* **55B** 224
Batalin J A and Vilkovisky G A 1977 *Phys. Lett.* **69B** 309
Fradkin F S and Fradkina T E 1978 *Phys. Lett.* **72B** 343
[14] Monaghan S 1986 *Phys. Lett.* **178B** 231
[15] Neveu A and West P C 1986 *Phys. Lett.* **182B** 343
[16] Casalbuoni R 1987 Talk given at *XII Edicion de los Encuentros Relativistas (La Laguna, Spain)*
[17] Barducci A, Casalbuoni R, Dominici D and Gatto R 1987 *Phys. Lett.* **187B** 135-40; 1987 *Phys. Lett.* **194B** 257
[18] Siegel W and Zwiebach B 1987 *Nucl. Phys.* B **282** 125
[19] Bars I and Kallosh R E 1989 *Phys. Lett.* **233B** 117
[20] Hull C M, Spence W J, Vazquez-Bello J L 1991 *Nucl. Phys.* **348B** 108
[21] Gunaydin M, Nilsson B E W, Sierra G and Townsend P K 1986 *Phys. Lett.* **176B** 45
[22] Aratyn H, Ingermanson R and Niemi A J 1987 *Phys. Lett.* **189B** 427
Niemi A J 1987 Lie algebra cohomology and the $Osp(1, 1/2)$ structure string theory *Preprint* UCB-PTH-87/29
[23] van Holten J W 1989 *Phys. Lett.* **B216** 133
[24] Hartley D H 1988 *PhD Thesis* St Andrews University
[25] Hartley D H and Cornwell J F 1988 *J. Phys. A: Math. Gen.* **21** 3171
[26] Cornwell J F 1989 *Supersymmetries and Infinite Dimensional Algebras (Group Theory in Physics 3)* (New York: Academic)
[27] Parker M 1980 *J. Math. Phys.* **21** 689
[28] Govaerts J 1991 *Hamiltonian Quantisation and Constrained Dynamics (Leuven Notes in Mathematical and Theoretical Physics 4)* (Leuven: Leuven University Press)
[29] Govaerts J 1989 *Int. J. Mod. Phys.* A **4** 173

[30] Govaerts J 1989 *Int. J. Mod. Phys.* A **4** 4487
[31] Blattner R J 1969 *Trans. Am. Math. Soc.* **144** 457
[32] Henneaux M 1986 *Phys. Lett.* **177B** 35
[33] Henneaux M and Teitelboim C 1982 *Ann. Phys.* **143** 127
Teitelboim C 1982 *Phys. Rev.* D **25** 3159; 1982 *Quantization of Gauge Systems* (Princeton, NJ: Princeton University Press)
[34] Bardakci K and Halpern M B 1968 *Phys. Rev.* **176** 1686
[35] McClain B, Niemi A J, Taylor C and Wijewardhana 1983 *Nucl. Phys.* B **217** 430;
[36] Ricotta R M 1989 Negative dimensions in quantum field theory *Sao Paulo preprint* IFT/P-33/89
[37] Kugo T and Ojima I 1979 Prog. Theor. Phys. Suppl. **66** 1
[38] Jarvis P D 1990 *J. Math. Phys.* **31** 1783
[39] Bowick M J and Gürsey F, 1986 *Phys. Lett.* **175B** 182
[40] van Holten J W 1990 *Nucl. Phys.* B **339** 158